Diana Davis   Math 34   January 29, 2018   Class #4 (1)
§2.1

Mathematician spotlight: Ralph Gomez, Associate Professor, Swarthmore
- differential geometry

Last time: cylindrical & spherical coordinates; graphing surfaces

This time: "quadric surfaces," and how to figure out what they look like.

In 2D, consider curves of the form
$ax^2+bxy+cy^2+dx+ey+f=0.$
Such an equation describes one of the following, depending on a,b,c,d,e,f:

________ ________

________ ________

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In 3D, consider surfaces of the form
$ax^2+by^2+cz^2+dxy+eyz+fxz+gx+hy+iz+j=0.$
What do these surfaces look like?

Idea: Set $x=$ constant, or $y=$ constant, etc.
This gives us a cross section (level curve) and reduces it to the 2D case of ellipses, parabolas, etc.!
Then use the cross sections to see what the surface is.

Example. $z=x^2-y^2$. We've seen this before! In color.
Let's look at cross-sections: Let K be a constant.

(1) $z=K \Rightarrow K=x^2-y^2$: family of hyperbolas
(2) $x=K \Rightarrow z=K^2-y^2$: family of downward-facing parabolas
(3) $y=K \Rightarrow z=x^2-K^2$: " " upward - " "

Conclusion: horizontal cross-sections are hyperbolas
vertical cross-sections in one direction look like ∪
and in the perpendicular direction look like ∩.

Note! $\frac{z}{c}=\frac{y^2}{a^2}-\frac{x^2}{b^2}$ is essentially the same as above, but stretched in each direction.
It's a "HYPERBOLIC PARABOLOID."
↳ because cross sections are hyperbolas & paraboloids,

Example. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

(1) $x=0 \Rightarrow$ ellipse in yz-plane
(2) $y=0 \Rightarrow$ " " xz-plane
(3) $z=0 \Rightarrow$ " " xy-plane

What if $x=a$? ________________

What if $x=2a$? ________________

Cross sections are ellipses, a single point, or empty.

ELLIPSOID

Example. $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

$x=0$: $z=y^2$ parabola
$y=0$: $z=x^2$ " "
$z=0$: point $(0,0,0)$
$z>0$: ellipse
$z<0$: empty
$x=K$: parabola shifted up
$y=K$: " " "

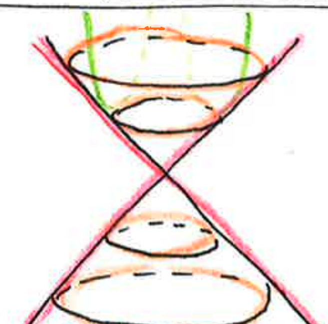
ELLIPTIC PARABOLOID

Now we will consider the family of surfaces of the form $\boxed{z^2 = x^2 + y^2 \pm K.}$

First, suppose $K=0$.

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- $x=0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y$ 
- $y=0 \Rightarrow z^2 = x^2 \Rightarrow z = \pm x$ "
- $z=0 \Rightarrow x=0, y=0 \Rightarrow (0,0,0)$
- $z=k \Rightarrow k^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow$ ellipse 
- $x=k, y=k \Rightarrow$ upward parabolas 

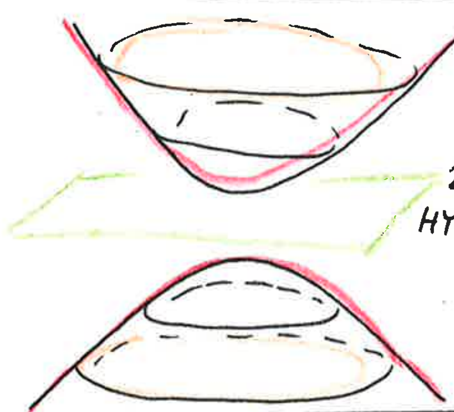


ELLIPTIC CONE

Now, suppose $K > 0$ (added to x^2+y^2).

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + K$$

- $x=0$: $z^2 = y^2 + K$, hyperbola
- $y=0$: $z^2 = x^2 + K$, " "
- $z=0$: EMPTY! tells you that the surface doesn't intersect the xy-plane! So 2 pieces!
- $z^2 > K$: $(+) = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow$ ellipse

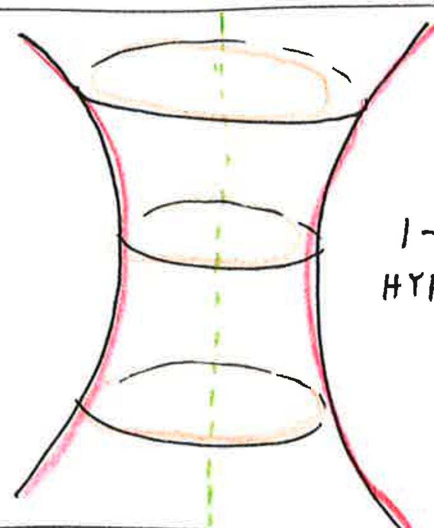


2-SHEETED HYPERBOLOID

Now, suppose $K < 0$ (subtracted from x^2+y^2).

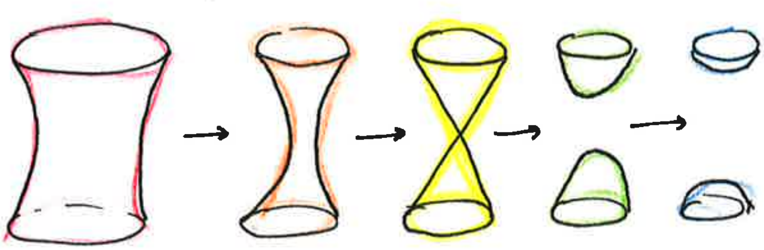
$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} - K$$

- $x=0$: $z^2 = y^2 - K$, hyperbola
- $y=0$: $z^2 = x^2 - K$, "
- $z=0$: $K = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, ellipse
- $(x,y) = (0,0) \Rightarrow z^2 = -K$ NO SOLUTION tells you that the surface doesn't intersect the z-axis
- $z=c \Rightarrow c^2 + K = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow$ ellipse



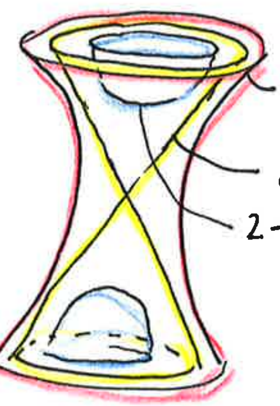
1-SHEETED HYPERBOLOID

Finally, think about the surfaces as K goes from negative, to 0, to positive, like a movie:



$K = -$big $K = -$small $K = 0$ $K = +$small $K = +$big

Or, think of them as nested:



1-sheeted hyperboloid

cone

2-sheeted hyperboloid

Just as hyperbolas approach asymptote lines: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 \rightarrow y = \pm \frac{b}{a} x$, Hyperboloids approach asymptote cones: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \cancel{\pm 1}$