Diana Davis Math 34 January 26, 2018 Class #3 ① §1.7

Mathematician spotlight: Moon Duchin, associate professor at Tufts University.

- geometric group theory, geometric topology, dynamical systems.
- gerrymandering: using geometry to advance civil rights.

Last time: introduction to functions of two & three variables, $f(x,y)$ and $f(x,y,z)$.

Today: cylindrical & spherical coordinates.

- they work great for round surfaces like cylinders & spheres (!), cones, ...
- you can easily describe some surfaces that are not graphs of functions, $z = f(x,y)$.

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① Cylindrical coordinates (r, θ, z). "Polar with z."

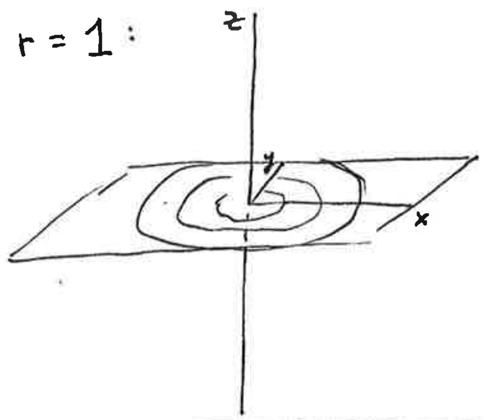
distance from z-axis ↗ r

ccw angle from positive x-axis ↗ θ

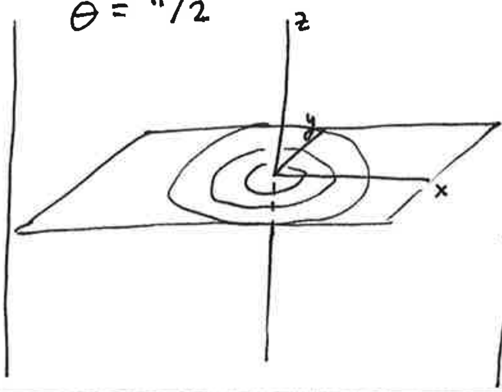
height above xy-plane ↗ z

OK, so we have 3 variables. If we fix one, and let the other two be anything, we get a surface. Let's see what we get.

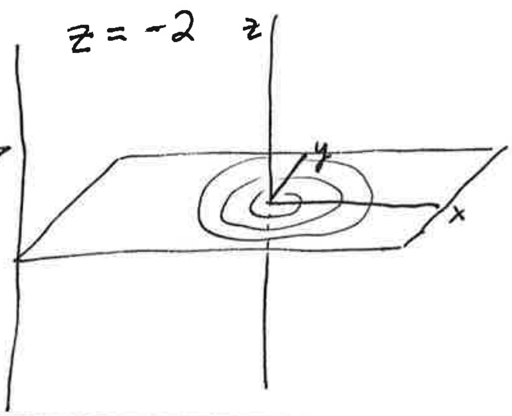
$r = 1$:



$\theta = \pi/2$

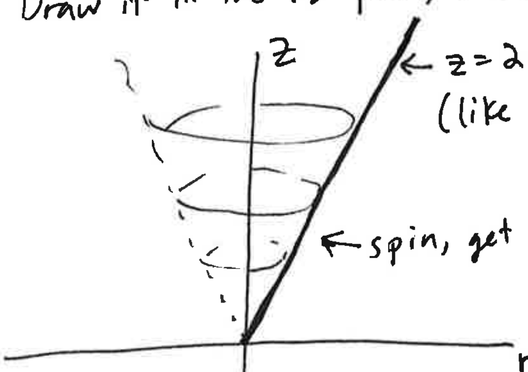


$z = -2$



Example. $z = 2r$.

Note that θ can be anything, so the surface has rotational symmetry. Draw it in the rz-plane, then spin!



← $z = 2r$ (like $y = 2x$)

← spin, get cone!

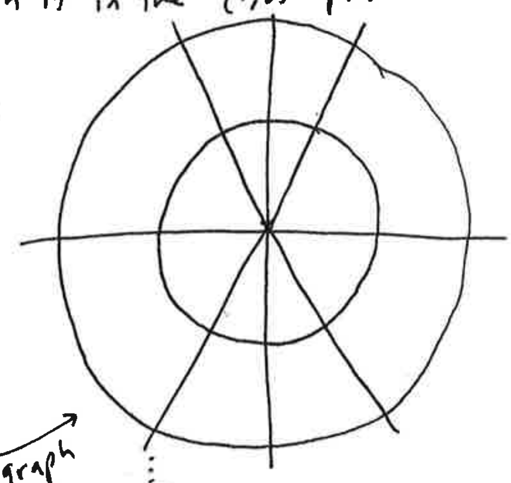
Let's assume $r \geq 0$.

r (represents side view of xy-plane)

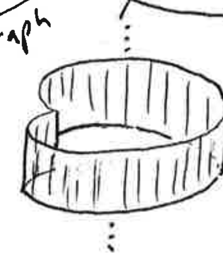
Example. $r = 1 + \cos\theta$.

Note that z can be anything, so it's a vertical "cylinder" (not necessarily circular) over some shape. Let's graph it in the (r,θ)-plane.

θ	$\cos\theta$	$1+\cos\theta$
0	1	
$\pi/3$	$1/2$	
$\pi/2$	0	
$2\pi/3$	$-1/2$	
π	-1	
$4\pi/3$	$-1/2$	
$3\pi/2$	0	
$5\pi/3$	$1/2$	
2π	1	



graph



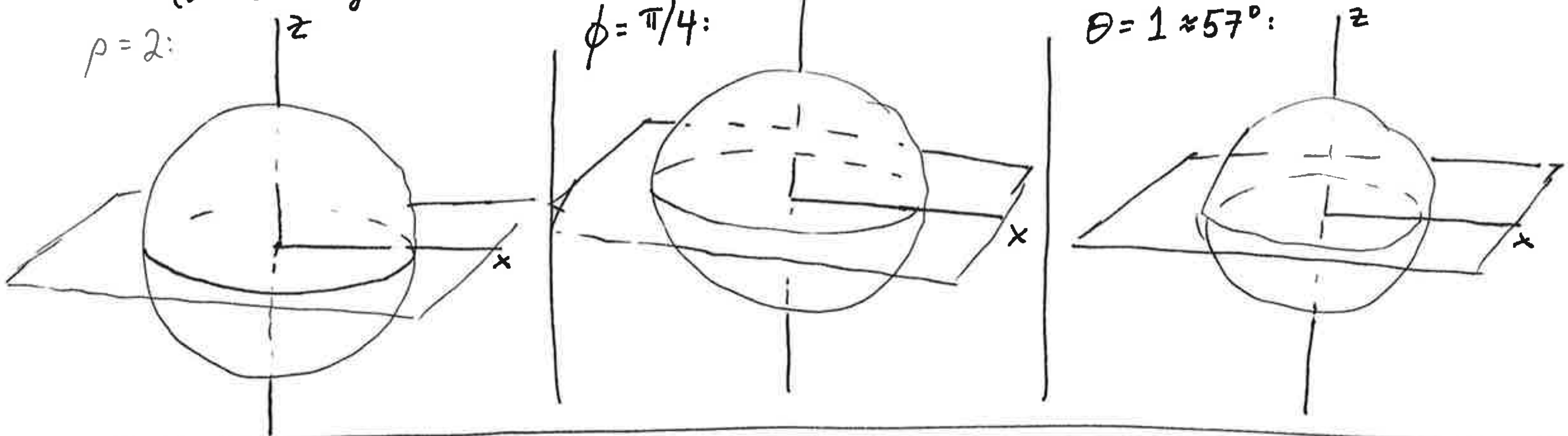
cylinder over a cardioid

② Spherical coordinates (ρ, φ, θ)

- ρ: distance (radius) from origin
- φ: angle down from positive z-axis
- θ: angle ccw from positive x-axis (same as cylindrical).

← Some books, and most physicists, use the order (ρ, θ, φ). For clarity, we will always write $(\rho, \varphi, \theta) = \ldots$ so there is no confusion.

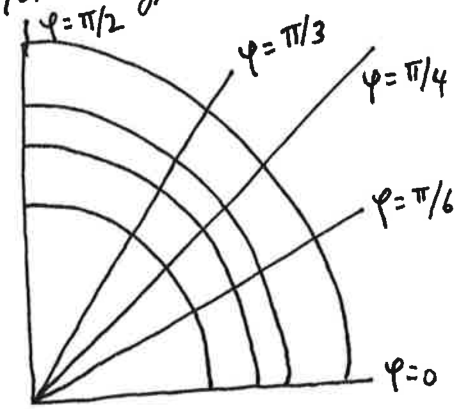
We usually restrict to $\rho \geq 0$.



__Example.__ $\rho = 2a\cos\varphi$.

Note that θ is not in the equation, so θ can be anything, so the surface must be rotationally symmetric. Equivalently, cross sections for each value of θ look the same.

φ	$\cos\varphi$	$2a\cos\varphi$
0	1	$2a$
$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}\,a$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}\,a$
$\pi/3$	$1/2$	a
$\pi/2$	0	0

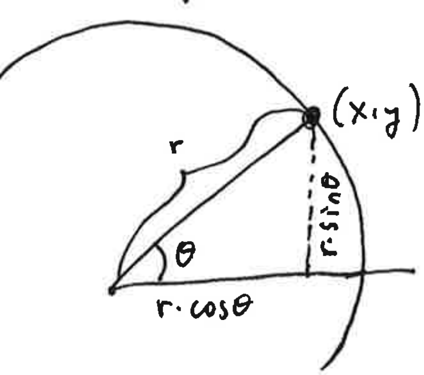


Sketch the surface in $\mathbb{R}^3$:

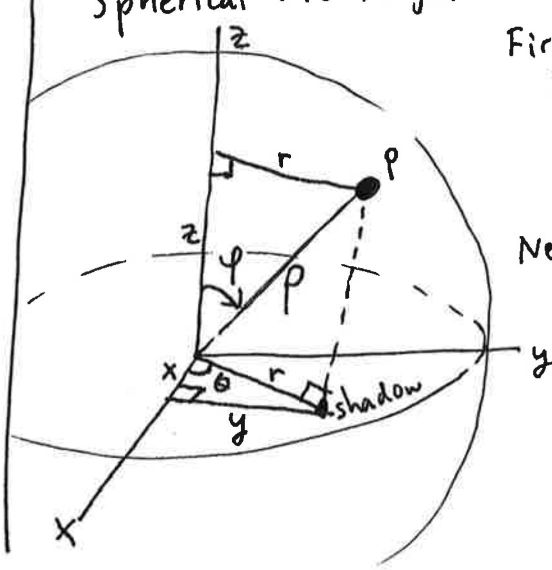
Going back and forth between cylindrical & spherical, to rectangular coordinates:

Cylindrical → rectangular

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z \end{cases}$$



Spherical → rectangular



First step: use "shadow" of P in the xy-plane.

$x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$ } as in polar & cylindrical.

Next step: use vertical triangle.

$z = \rho\cdot\cos\varphi$ ←

$r = \rho\cdot\sin\varphi$

$\Rightarrow x = \rho\cdot\sin\varphi\cdot\cos\theta$ ←

$\Rightarrow y = \rho\cdot\sin\varphi\cdot\sin\theta$ ←

☺