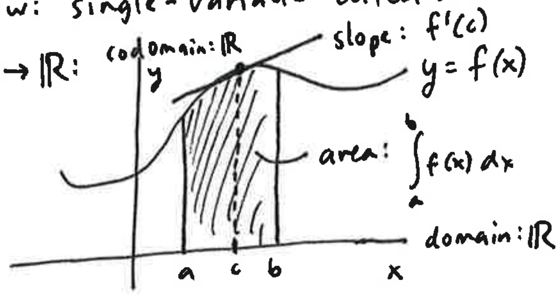
Mathematician spotlight: John Urschel – PhD student at MIT, Penn St undergrad.
- spectral graph theory, numerical linear algebra, machine learning
- Baltimore Ravens, guard, 2014-2017.

You know: single-variable calculus.

$f: \mathbb{R} \to \mathbb{R}$:

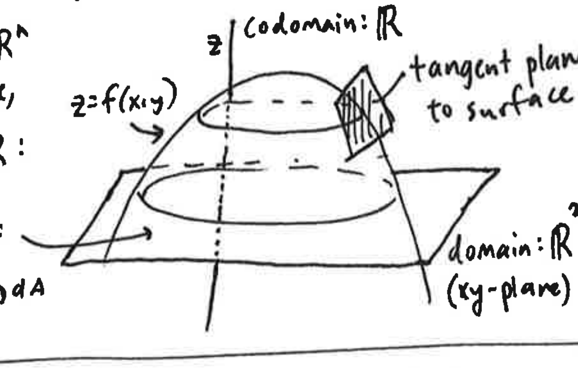


You want to learn: multivariable calculus.

$f: \mathbb{R}^m \to \mathbb{R}^n$

for example,

$f: \mathbb{R}^2 \to \mathbb{R}$:



We'll study functions of several variables. Here is some vocabulary to talk about them:

<u>domain:</u> all possible inputs

<u>codomain:</u> the set where outputs live

<u>range:</u> outputs that are actually achieved

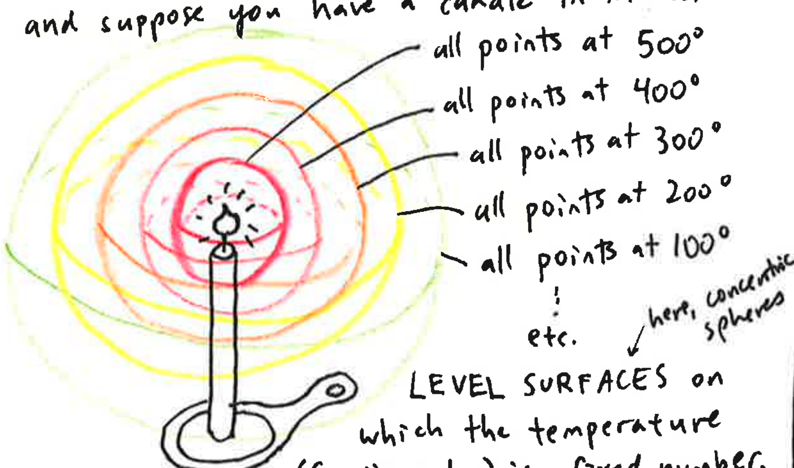
A function is <u>onto</u> if its range is the entire codomain.

A function is one-to-one if each element of the range comes from exactly one element in the domain, i.e. $f(x)=f(y) \Rightarrow x=y$.

function	domain	codomain	range	onto?	one-to-one?
$f(x)=x^2$	$\mathbb{R}$	$\mathbb{R}$	nonnegative real #s.	no, misses negatives	no, $f(-1)=f(1)$.
$f(x,y)=x+y+z$					
f(point in room) = temperature					
f(point on field) = wind direction					

To visualize multivariable functions, use "level curves" and "level surfaces."

Example. Let $f(x,y,z) =$ temperature at point (x,y,z) and suppose you have a candle in the cold air.



LEVEL SURFACES on which the temperature (function value) is a fixed number.

Example. Let $f(x,y,z) = x+y+z$.

What does this function look like?

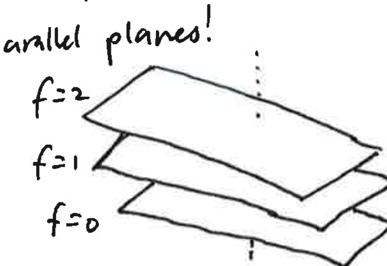
Let's try some fixed function values.

$f=0 \Rightarrow x+y+z=0$, plane through $(0,0,0)$ w/ $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$f=1 \Rightarrow x+y+z=1$, plane through $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ w/ $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$f=2 \Rightarrow x+y+z=2$, plane through $(0,0,2)$, etc. w/ $\vec{n}=$

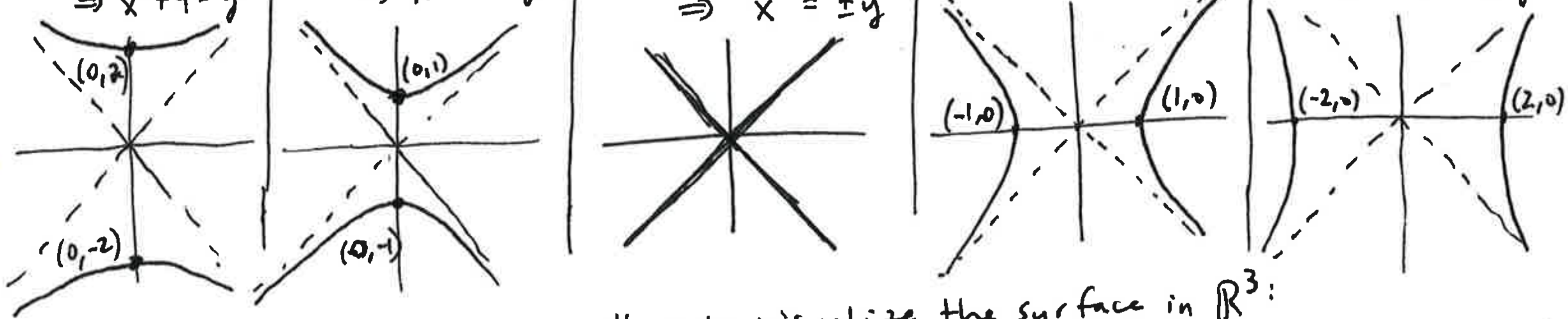
these are parallel planes!



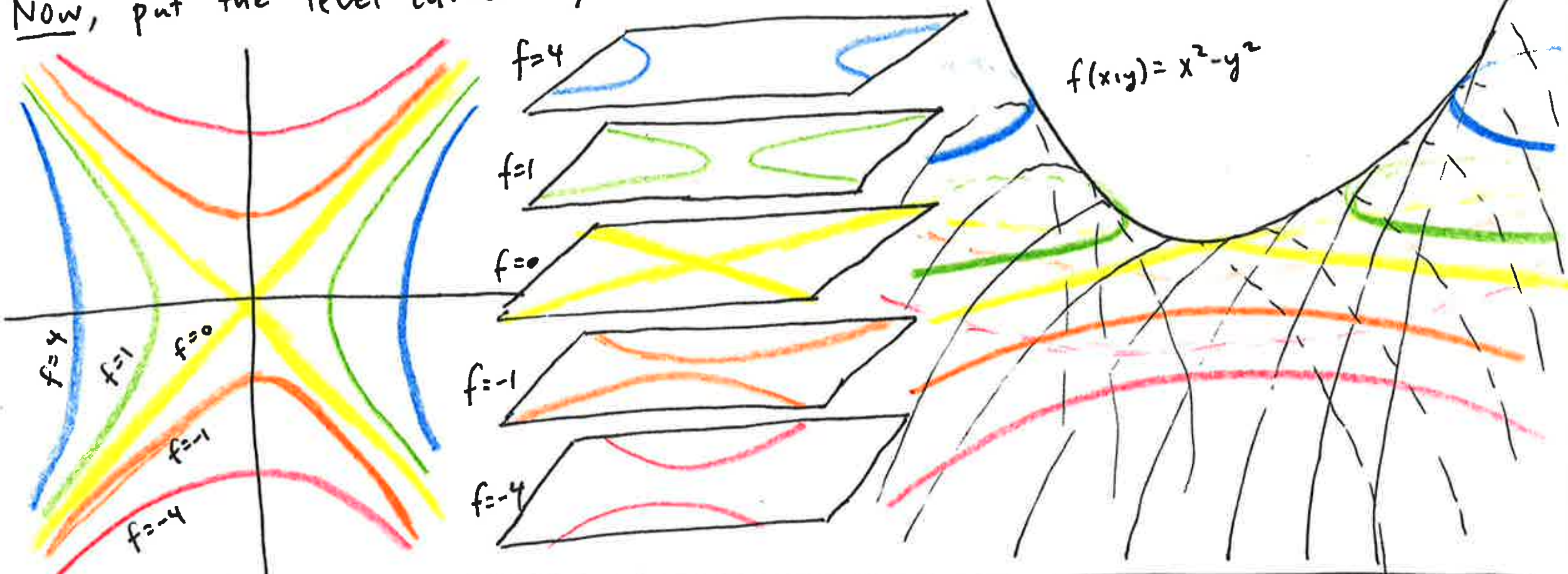
Those were level <u>surfaces</u> for functions $f: \mathbb{R}^3 \to \mathbb{R}$ that we can't draw.
We can use level <u>curves</u> for functions $f: \mathbb{R}^2 \to \mathbb{R}$ to understand and draw them.

<u>Example.</u> Sketch level curves at levels $-4, -1, 0, 1, 4$ for $f(x,y) = x^2 - y^2$.

$f=-4 \Rightarrow x^2-y^2=-4$	$f=-1 \Rightarrow x^2-y^2=-1$	$f=0 \Rightarrow x^2-y^2=0$	$f=1 \Rightarrow x^2-y^2=1$	$f=4 \Rightarrow x^2-y^2=4$
$\Rightarrow x^2+4=y^2$	$\Rightarrow x^2+1=y^2$	$\Rightarrow x^2=y^2$	$\Rightarrow x^2=1+y^2$	$\Rightarrow x^2=4+y^2$
		$\Rightarrow x=\pm y$		



<u>Now</u>, put the level curves together to visualize the surface in $\mathbb{R}^3$:



<u>Definition.</u> Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a $\overbrace{\text{scalar-valued}}^{\text{codomain is } \mathbb{R}}$ function of $\overbrace{\text{two variables}}^{\text{domain is } \mathbb{R}^2}$.
Then the <u>level curve of f at height c</u> is the curve in $\mathbb{R}^2$ defined by $f(x,y)=c$, i.e. the set of points $\{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$.

Similarly, if $g: \mathbb{R}^3 \to \mathbb{R}$ is a scalar-valued function of three variables, the <u>level surface of g at level c</u> is the surface $\{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = c\}$.

<u>Remark.</u> Some surfaces in $\mathbb{R}^3$ cannot be described as a function $f: \mathbb{R}^2 \to \mathbb{R}$, i.e. $z = f(x,y)$, because they are not graphs of functions, because they fail the "vertical line test."

<u>Example.</u> Sphere $x^2+y^2+z^2=r^2$