Diana Davis Math 34 January 22, 2018 Class #1 (1) §1.1–1.5

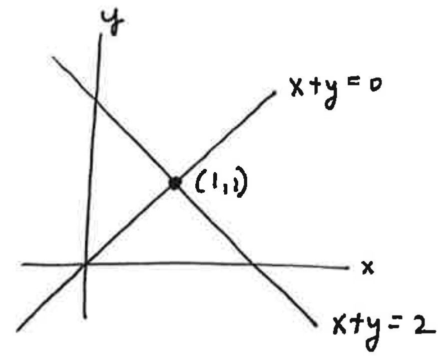
Mathematician spotlight: Diana Davis (my introduction).

Overview of course and syllabus:
- Come to class! 11:30 class will be videotaped.
- Homework: WebWork (due Mon) & written homework (due Fri).
- Exams: Friday 23 Feb, Friday 6 April, Exam week.
- Materials: 3-ring binder (for these notes), colored pencils/pens.
- Textbook: Susan Colley, Vector Calculus, 4th edition.

Today: lines, planes and the cross product.

In linear algebra, you solved systems of linear equations.

Example. Find the point where two lines intersect:



$$\begin{cases} x+y=2 \\ x-y=0 \end{cases} \xrightarrow[\text{matrix}]{\text{in}} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x=1 \\ y=1 \end{array}$$

[Two variables] − [Two equations] = [zero degrees of freedom] ⇒ solution set is empty or is isolated point(s).

In this case, solution is one point.

Question. What kind of object does $x-2y+3z=6$ describe?
- It's a linear equation (no x^2, no $\sin(y)$, etc.) so it's a line, plane, etc. (not curved)
- [THREE variables] − [ONE equation] = [Two deg. of freedom] ⇒ it's a plane!

- Give some examples of points on this plane: ______

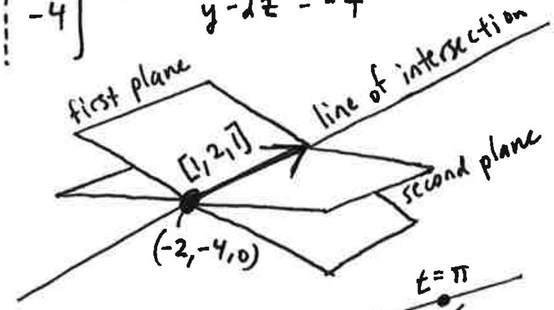
Question. How would we write the equation of a line in $\mathbb{R}^3$?

Multi-tasking. Let's find the line that is the intersection of two $\overbrace{\text{non-parallel}}^{\text{we'll explore this part soon.}}$ planes.

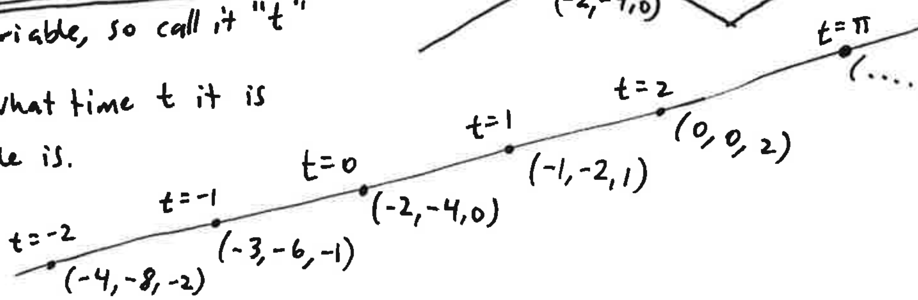
$$\begin{cases} x-2y+3z=6 & \leftarrow \text{first plane} \\ 3x-2y+z=2 & \leftarrow \text{second plane} \end{cases} \xrightarrow[\text{matrix}]{\text{in}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 3 & -2 & 1 & 2 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -4 \end{array}\right] \Rightarrow \begin{array}{l} x \quad -z=-2 \\ y-2z=-4 \end{array}$$

$$\Rightarrow \begin{cases} x=-2+z \\ y=-4+2z \\ z=0+z \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} t.$$

↖ z is our free variable, so call it "t"

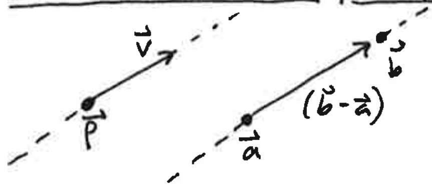


This is a parametric equation: what time t it is tells you where $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ your particle is.



To define a line, you need: • a point $\vec{p}$ and a direction $\vec{v}$: $\vec{r}(t) = \vec{p} + t\vec{v}$

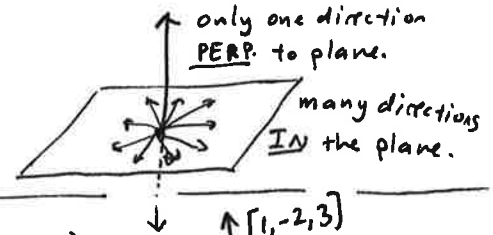
• two points $\vec{a}$ and $\vec{b}$: $\vec{r}(t) = \vec{a} + (\vec{b}-\vec{a})t$



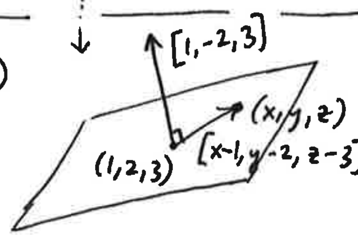
To define a plane, you need: • three points

• a point and a PERPENDICULAR! direction

⋮ more options



Goal: Write an equation for the plane containing the point $(1,2,3)$ that has $[1,-2,3]$ as its normal vector.



Idea: Notice that for any point (x,y,z) on the plane,

$$[1,-2,3]\cdot[x-1, y-2, z-3] = 0 \Rightarrow 1(x-1) - 2(y-2) + 3(z-3) = 0$$

$$\Rightarrow x - 1 - 2y + 4 + 3z - 9 = 0 \Rightarrow \underline{x - 2y + 3z = 6.}$$

In general: $[a,b,c]\cdot[x-x_0, y-y_0, z-z_0] = 0$

$$\Rightarrow \underbrace{ax+by+cz}_{\text{normal vector gives you coefficients}} = d = \underbrace{ax_0+by_0+cz_0.}$$ ← to get constant, plug in a point.

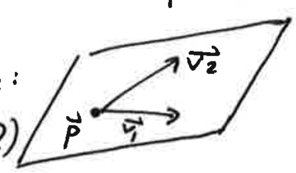
What if we had used $(0,0,2)$ as our point instead? $1(x-0) - 2(y-0) + 3(z-2) = \underline{x-2y+3z=6.}$ as before.

Okay, but what if I'm not given the normal vector (perpendicular direction) to the plane?

What if I have: • three points in the plane

• a point on the plane and two directions in the plane:

(these amount to the same information, do you see why?)



We need: A method to take two (non-parallel) vectors in $\mathbb{R}^3$, and find a new vector that is perpendicular to the first two.

$[x_1,y_1,z_1]$
$[x_2,y_2,z_2]$
↓
$[a,b,c]$

Solve this: $\begin{cases}[a,b,c]\cdot[x_1,y_1,z_1] = 0 \\ [a,b,c]\cdot[x_2,y_2,z_2] = 0\end{cases} \Rightarrow$ 3 variables a,b,c; 2 equations $\Rightarrow$ 1 degree of freedom, length of vector.

Skip to answer: $\underset{\text{vector perp. to both}}{[a,b,c]} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \vec{i}(y_1z_2 - z_1y_2) - \vec{j}(x_1z_2 - z_1x_2) + \vec{k}(x_1y_2 - y_1x_2)$

Cross product! determinant → $= [y_1z_2 - z_1x_2,\ x_1z_2 - z_1x_2,\ x_1y_2 - y_1x_2].$

Example. Given 3 points:

$(0,0,2)$
$(0,-3,0)$
$(6,0,0)$

$[0,3,2]$, $[6,3,0]$

$$[0,3,2]\times[6,3,0] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 6 & 3 & 0 \end{vmatrix}$$

$= [-6, 12, -18]$ or reduce to $[1,-2,3]$. ◡̈

Length of cross product vector:

Area of parallelogram spanned by the two input vectors.

Direction: follows right hand rule.

fingers toward $\vec{v_1}$, curl toward $\vec{v_2}$ $\Rightarrow$ thumb points in direction $\vec{v_1}\times\vec{v_2}$.